Chapter 5 and 6

Section 2: Subspaces

(book sections 5.1 and 6.2)

Ideas in this section…

1) Definition of a subspace of a vector space
2) How to prove that a subset of a vector space IS a subspace
3) How to prove that a subset of a vector space IS NOT a subspace

Subspace of a Vector Space

Def: If V is a vector space, a nonempty subset $U \subseteq V$ is a subspace of V if U is itself a vector space under the addition and scalar multiplication rule inherited from V.

Note: This means that U satisfies the 10 vector space properties.

Thm 6.2.1 **Subspace Test**: A subset U of a vector space V is a subspace if
1) $U \neq \emptyset$
2) U is closed under the vector addition rule of V
3) U is closed under the scalar multiplication rule of V

Subspace of a Vector Space

Axioms for vector addition modify to U

A1. *If* $\mathbf{u}$ *and* $\mathbf{v}$ *are in* V, *then* $\mathbf{u}+\mathbf{v}$ *is in* V.

A2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ *for all* $\mathbf{u}$ *and* $\mathbf{v}$ *in* V.

A3. $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$ *for all* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *in* V.

A4. *An element* $\mathbf{0}$ *in* V *exists such that* $\mathbf{v}+\mathbf{0}=\mathbf{v}=\mathbf{0}+\mathbf{v}$ *for every* $\mathbf{v}$ *in* V.

A5. *For each* $\mathbf{v}$ *in* V, *an element* $-\mathbf{v}$ *in* V *exists such that* $-\mathbf{v}+\mathbf{v}=\mathbf{0}$ *and* $\mathbf{v}+(-\mathbf{v})=\mathbf{0}$.

Axioms for scalar multiplication

S1. *If* $\mathbf{v}$ *is in* V, *then* $a\mathbf{v}$ *is in* V *for all* a *in* $\mathbb{R}$.

S2. $a(\mathbf{v}+\mathbf{w})=a\mathbf{v}+a\mathbf{w}$ *for all* $\mathbf{v}$ *and* $\mathbf{w}$ *in* V *and all* a *in* $\mathbb{R}$.

S3. $(a+b)\mathbf{v}=a\mathbf{v}+b\mathbf{v}$ *for all* $\mathbf{v}$ *in* V *and all* a *and* b *in* $\mathbb{R}$.

S4. $a(b\mathbf{v})=(ab)\mathbf{v}$ *for all* $\mathbf{v}$ *in* V *and all* a *and* b *in* $\mathbb{R}$.

S5. $1\mathbf{v}=\mathbf{v}$ *for all* $\mathbf{v}$ *in* V.

Once A1 and S1 are proved for U, all others follow bec. $U \subseteq V$ except A4 and A5

Subspace of a Vector Space

<u>Thm 6.2.1 **Subspace Test**</u>: A subset U of a vector space V is a subspace if
1) $U \neq \emptyset$
2) U is closed under the vector addition rule of V
3) U is closed under the scalar multiplication rule of V

A4. An element $\mathbf{0}$ *in* V *exists such that* $\mathbf{v} + \mathbf{0} = \mathbf{v} = \mathbf{0} + \mathbf{v}$ *for every* $\mathbf{v}$ *in* V. modify to U

<u>Proof of A4:</u>

Subspace of a Vector Space

Thm 6.2.1 **Subspace Test**: A subset U of a vector space V is a subspace if
1) $U \neq \emptyset$
2) U is closed under the vector addition rule of V
3) U is closed under the scalar multiplication rule of V

modify to U

A5. For each $\mathbf{v}$ *in* V, *an element* $-\mathbf{v}$ *in* V *exists such that* $-\mathbf{v}+\mathbf{v}=\mathbf{0}$ *and* $\mathbf{v}+(-\mathbf{v})=\mathbf{0}$.

Proof of A5:

Subspace of a Vector Space

<u>Result</u>: If U is a subspace of V, then the zero vector $\vec{0}$ from V must be in U.

<u>Note</u>:

- So if the zero vector $\vec{0}$ from V is not in U, then U cannot be a subspace of V
- This gives you a quick way to check that U is not be a subspace of V
- But this doesn't work all the time

Subspace of a Vector Space

<u>Ex 1</u>: $\{\vec{0}\}$ and V are subspaces of V

Subspace of a Vector Space

Ex 2: If $\vec{v} \in V$, $U = \{ c\vec{v} \mid c \in \mathbb{R} \}$ is a subspace of V

Subspace of a Vector Space

Ex 3: Show that $U = \{\, (a, b, c, d) \mid a + b + c = 0,\ d = 2a,\ a, b, c, d \in \mathbb{R} \,\}$ is a subspace of $\mathbb{R}^4$

Subspace of a Vector Space

Ex 3: Show that $U = \{ (a, b, c, d) \mid a + b + c = 0,\ d = 2a,\ a, b, c, d \in \mathbb{R} \}$ is a subspace of $\mathbb{R}^4$

Ex 3 variation 1: $U = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \ a + b + c = 0,\ d = 2a,\ a, b, c, d \in \mathbb{R} \right\}$ is a subspace of M_{22}

Ex 3 variation 2: $U = \{ a + bx + cx^2 + dx^3 \mid a + b + c = 0,\ d = 2a,$ $a, b, c, d \in \mathbb{R} \}$ is a subspace of P_3

Subspace of a Vector Space

Ex 4: Show that $U = \{\, (x_1,\ 2x_1 - x_2,\ x_1 + x_2) \mid x_1\,, x_2 \in \mathbb{R}\,\}$ is a subspace of $\mathbb{R}^3$

Subspace of a Vector Space

<u>Ex 5</u>: Show that $U = \{ (t,\ t,\ t+1) \mid t \in \mathbb{R} \}$ is not a subspace of $\mathbb{R}^3$

Subspace of a Vector Space

Ex 6: Show that $U = \{ (t^2,\ 4t) \mid t \in \mathbb{R} \}$ is not a subspace of $\mathbb{R}^2$

Subspace of a Vector Space

Ex 7: Show that $U = \{\, (a,\ b) \mid 2a - 5b = 1,\ \ a, b \in \mathbb{R} \,\}$ is not a subspace of $\mathbb{R}^2$

Subspace of a Vector Space

Ex 8: Let A be a fixed matrix in M_{nn}. Show that $U = \{\, X \in M_{nn} \mid AX = XA \,\}$ is a subspace of M_{nn}

Subspace of a Vector Space

Ex 9: Consider the set U of all polynomials that have the number 3 as a root.

$$U = \{ \, p(x) \in P \mid p(3) = 0 \, \}$$

Show that U is a subspace of P

Subspace of a Vector Space

Ex 10: P_n is a subspace of P for each $n \geq 0$ Nothing to show

Subspace of a Vector Space

Ex 11: Show that $D[a,b]$ which is the set of all differentiable functions defined on $[a,b]$ is a subspace of the vector space $F[a,b]$ of all functions defined on $[a,b]$

What you need to know from the book

Book reading

Section 5.1 pages 263 - 264
Section 6.2 pages 338 - 341

Problems you need to know how to do from the book

Section 5.1 page 269 #'s 1, 6, 15, 16abce, 17, 19, 20
Section 6.2 page 343 #'s 1 - 5, 22 - 23